



Solutions To Plate Problem With Constant Rigidity Using Singularity Functions

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Abstract— A new and novel method of solving problems of rectangular plates of constant stiffness using singularity functions is proposed. The working methods and manifestations of the use of singularity functions in plate analysis are presented.

Use of singularity functions, to represent the loading and reactive forces and moments at the boundary, along with the finite difference equations for plates of constant stiffness, is the new direction of plate analysis proposed. Here for some problems of plates of constant stiffness, the method is to find mathematical solution to the basic

partial differential equations (namely, $\nabla^4 w = \frac{q}{D}$) along

with the representation of the load q in terms of singularity function to get a closed form solution is presented.

The advantage of the proposed method over finite elements lies in the fact, that the solution can be found without the help of computers, In contrast, the need for a computer for even a simple problem in the finite element method is required.

The solution of certain types of problems of plates with constant and variable stiffness for certain types of loading and boundary conditions is included to exemplify the technique of use of singularity functions.

KEY WORDS-Singularity function, finite difference method.

NOTATIONS:

∇ Nabla (or del operator)

w Deflection function

q Loading function

D Flexural rigidity of plate

E Modulus of Elasticity

μ Poisson's ratio

M_x, M_y Bending Moments along x and y direction

V_x, V_y Shear Forces along x and y direction

h Spacing size (mesh size)

M_{xy}, M_{yz} Torsional Moments

$\delta(x), \delta(y)$ Dirac Delta Functions

$U(x), U(y)$ Unit Step Functions

INTRODUCTION

In the analysis and design of plates, the evaluation of elastic functions such as moment, deflection, stress and strains is of supreme importance. The concerned designer finds some practical difficulties in designing plates owing to its tedious calculations involved in its analysis. There is a constant search by engineers to find a simpler method of an analysis to get insight into these problems

Due to the complexity in solving problems of plates, only certain limited methods are available for determining the two important basic quantities, namely slope and deflection. For any type of problem, it had been a customary to employ direct approach i.e., classical approach. Classical methods involve derivation of certain expression containing several parameters governing the plates. Plates with simple geometry and simple loading conditions, the above stated method may seem to be easy and result yielding. But for plates with complex geometry and loading condition, it may not converge to a solution. Many times, the setting up of concerned equations itself create mind boggling situations.

With the advent of computers, numerical methods to solve plate problems have found a key position in the analysis of plates and also in the calculation of deflection and slopes. Analyst agrees that the solutions obtained either by finite element method or by finite difference method are not exact, but approximate within permissible limits. The exactness of result can be increased by

increasing the number of elements or divisions, respectively.

As every method has its own limitation, these limitations limit the usefulness of the method. Time and again researchers try to devise new methods to overcome limitations. An attempt in that direction is use of singularity functions which enables (mathematically) to take of the abrupt changes in geometry and loading condition to analyse plate problems.

Some uses of singularity function in structural analysis have been indicated by many authors. Till now they have incorporated singularity function in analysis of beams. In this study, the potentialities of singularity functions is amplified and exploited successfully for the analysis of plates. The method in principle involves in the use of

basic plate equation $\nabla^4 w = \frac{q}{D}$ of plates. The L.H.S.

$\nabla^4 w$ is expressed in the form of finite difference equations (to avoid integration of fourth order partial differential equation) and the R.H.S. namely q and D are expressed using singularity functions.

Although, Levy's and Navier's solution give closed form solution (e.g. Deflection, at any desired point on plate), the labour involved in analysis is more. If finite difference method with singularity function is adopted, one can find deflection at sufficient number of points and hence the need for going to cumbersome methods is avoided.

The proposed technique of analysis of plates can be adapted, in general to any type of boundary, loading, stiffness variation of rectangular plates. Analysis can be made with or without the use of computers.

1.2 CHARACTERISTICS OF SINGULARITY FUNCTION:

Singularity function has been envisaged as an important tool to handle the points of discontinuity. More conveniently it can be stated as functions used to bring in linkage at points of physical discontinuity, mathematically.

In contrast to the classical approaches already in practice, to the problems of thin plates, the method of singularity functions is useful and superior when

- There are complicated load systems on the plate
- The plate is of variable stiffness
(I.e. stepped, continuous or combination of two types of variation of moment of inertia. Or I is variable i.e. I -Moment of Inertia).
- The plate is of composite nature (i.e. made of two or more types of materials or E is variable i.e. E -Modulus of Elasticity).
- Quantities like slope and deflection are to be evaluated at number of points in either direction, including the position and magnitude of maximum deflection or equation to elastic curve is required.
- There is a combination of above mentioned aspects.

- Values of M_{xy}, M_x, M_y -the support moments
- To avoid computers for computation, even though they are amenable to computer application.

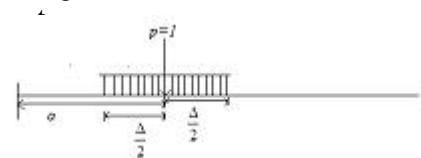
Since the singularity functions (or pathological functions as they are also called) do not have properties of differentiability in mathematical terms, they require some amount of caution while incorporating and using them.

The two singularity functions considered are

1.21 DIRAC DELTA FUNCTION:

This function enables to express a point force or any other such concentration in a convenient and useful form.

Fig 1:



the intensity of loading for the unit force can be formed, since

$$w(x, a) = Lt_{\Delta \rightarrow 0} \begin{cases} 0 & \text{when } x < (a - \Delta / 2) \\ 1/\Delta & \text{when } (a - \Delta / 2) < x < (a + \Delta / 2) \\ 0 & \text{when } x > (a + \Delta / 2) \end{cases}$$

The notation $w(x, a)$ indicates that there is a unit force at a distance "a" from the origin.

Considering the plate, in which the loading can be split up in two directions namely x and y respectively. Now the intensity of loading $w(x, a)$ over the x direction and $w(y, b)$ along the y direction

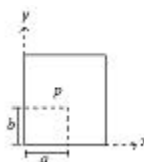
$$\int_0^{l_2} \int_0^{l_1} w(x, a) w(y, b) dx dy$$

case of two dimensional problems such as plate problem Dirac delta function can be used a

$$w(x, y) = P[\delta(x - a)\delta(y - b)]$$

to denote load p at (x, y) in a convenient and useful form.

Fig 2:



Load p at (a, b) distance.

Properties of Dirac Delta function:

- $\int \delta(x - a) dx = u(x - a) = \text{unit step function}$

$$2. \int g(x) \delta(x-a) dx = g(a)$$

Where 'a' indicates the location of singularity.

1.22. UNIT STEP FUNCTION:

This enables to express a loading function for the uniformly distributed distribution in a convenient and a useful form.

Properties of unit step function:

$$1. u(x-a) = \begin{cases} 0 & \text{when } x \leq a \\ 1 & \text{when } x > a \end{cases}$$

$U(x)$ becomes 1, for all positives values of x .

$$2. u(x-a) = [u(x-a)]^2 = \dots\dots\dots = [u(x-a)]^n$$

i.e. raising a step function to a power does not change the function.

$$3. [u(x-a_1)][u(x-a_2)] = u(x-a_2), \text{ if } a_2 > a_1$$

i.e., in a product of step functions, the step function farthest along the x axis takes over.

4. Integration of the product of a step function and some function $g(y)$ is

$$\int_{-\infty}^x g(y) u(x-a) dy = u(x-a) \int_a^x g(y) dy$$

$$5. \int_0^x (t-a)^n u(t-a) dt = u(x-a) \int_a^x (t-a)^n dt$$

$$6. (x-a)^n u(x-a) = \begin{cases} (x-a)^n & \text{when } n > 0 \text{ and } x > a \\ 0 & \text{when } n > 0 \text{ and } x < a \end{cases}$$

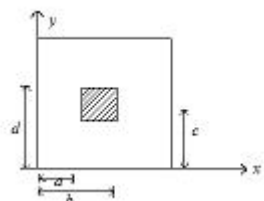
$$7. \frac{d}{dx} (x-a)^n u(x-a) = n(x-a)^{n-1} u(x-a) \text{ when } n > 1$$

$$8. \int (x-a)^n u(x-a) dx = \frac{(x-a)^{n+1}}{n+1} u(x-a) \text{ when } n > 0$$

1.3 Methodology

Working methods and innovations of using singularity function to two dimensional problems have been developed in this study and the results are encouraging.

Fig: 3(a)



The use of singularity functions in two dimensions proposed is

$w(x) = w_0 u(x-a) u(y-c) - w_0 u(x-b) u(y-d)$, w_0 is the rate of UDL. The first step function product $w_0 u(x-a) u(y-c)$ introduces the loading distribution of correct position (a, c) coordinate and the second step function product $w_0 u(x-b) u(y-d)$ cancels the contribution of the first expression at the proper position (b, d) and beyond it. The variation of modulus of elasticity E and moment of inertia I can be expressed in a very useful form using a unit step function.

Fig: 3(b) & 3(c)



$$\frac{1}{I_x} = \frac{1}{I_1} u(x) - \frac{1}{I_1} u(x-a) + \frac{1}{I_2} u(x-a) - \frac{1}{I_2} u(x-b) + \dots\dots\dots$$

$$\frac{1}{E_x} = \frac{1}{E_1} u(x) - \frac{1}{E_1} u(x-a) + \frac{1}{E_2} u(x-a) - \frac{1}{E_2} u(x-b) + \dots\dots\dots$$

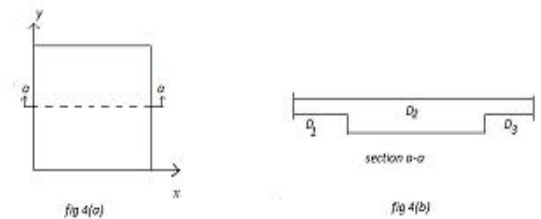
Where $0 < a < b$ and a and b indicates the location of singularity.

In problem of plates, we are interested in property of

flexural rigidity $D = \frac{Et^3}{12(1-\mu^2)}$ i.e., if E or plate

thickness t , or both of them changes flexural rigidity will change. It can change in both the directions namely 'x' and 'y' depending upon the cross section of the plate in either direction. In this case it can be demonstrated as follows:

Fig:



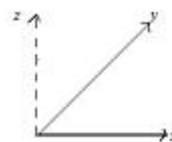
If variation of flexural rigidity is in 'x' direction, then $D_x = D_1 u(x) - D_1 u(x-a) + D_2 u(x-a) - D_2 u(x-b) + D_3 u(x-b) \dots\dots\dots$

Plate Equation using singularity functions:

The singularity functions are very useful in writing plate equations of constant (or variable) rigidity. The loading which is on R.H.S of plate equation viz., q & D expressed in terms of singularity function.

1. Sign convention:

Fig 5



Deflection upward is positive,
Load downward is positive.

2. Working methodology:

The plate equation for constant rigidity $\nabla^4 w = \frac{q(x,y)}{D}$ or

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q(x,y)}{D}$$

L.H.S. of the above equation is expressed in finite difference equations and R.H.S is expressed in terms of singularity functions.

After applying boundary conditions, we obtain a set of linear simultaneous equation which can be solved to obtain deflection and subsequently other elastic functions.

1.4 Certain plate problems having constant rigidity are worked to exemplify the new direction of analysis of plates.

Problem 1:

A square plate of size $a \times a$ is clamped along the opposite edges and simply supported at the other edges. Estimate the deflection at nodal points and maximum deflection (a is mesh size) Fig 6(a) & 6(b)

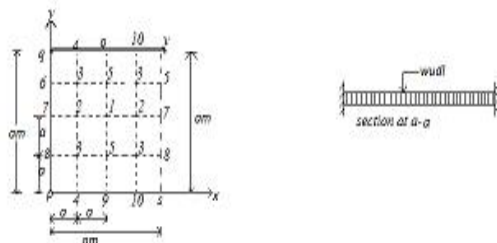


Plate with constant rigidity, Pq and rs fixed, ps and qr are simply supported and subjected to udl of w/m^2 .

Equation of plate with constant rigidity is

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q(x,y)}{D}$$

-(1)

Where w is deflection along Z axis
 q Is loading

$$D \text{ is flexural rigidity} = \frac{Et^3}{12} (1 - \mu^2)$$

In terms of singularity function and finite difference equation (1) can be written as

$$20w_{i,j} - 8(w_{i,j+1} + w_{i,j} + w_{i+1,j} + w_{i-1,j}) + 2(w_{i+1,j+1} + w_{i-1,j+1} + w_{i-1,j-1} + w_{i+1,j-1}) + (w_{i,j+2} + w_{i,j-2} + w_{i-2,j} + w_{i+2,j}) = \frac{q}{D} u(x)u(y)h^4$$

For Nodal point 1:

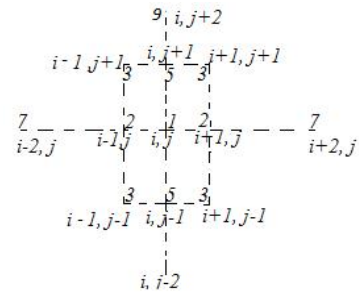


Plate equation in terms of finite difference and singularity function is

$$20w_1 - 8(2w_5 + 2w_2) + 2(4w_3) + 2(w_9 + w_7) = \frac{w_0}{D} u(x)u(y)h^4$$

or

$$20w_1 - 16w_5 - 16w_2 + 8w_3 = \frac{w_0}{D} u(x)u(y)h^4 \text{ as } w_9 = w_7 = 0$$

------(1a)

For any value of x , $u(x) = 1$

Here, $x = 2h, u(2h) = 1$ and

$y = 2h, u(y) = u(2h) = 1$

For Nodal point 2: $x = h, y = 2h$

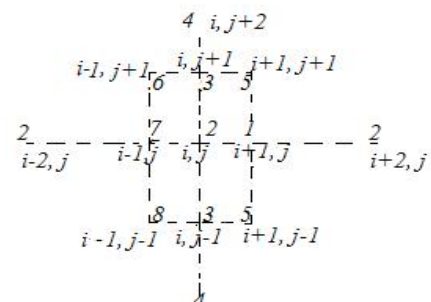


Plate equation in terms of finite difference and singularity function is

$$20w_2 - 8(w_3 + w_7 + w_1 + w_3) + 2(w_6 + w_5 + w_3 + w_8) +$$

$$(w_2 + w_4 + w_4 + w_2) = \frac{w_0}{D} u(x)u(y)h^4$$

$$\text{and } w_6 = w_8 = w_4 = 0$$

$$\text{or, } 20w_2 - 16w_3 - 8w_1 + 4w_5 + 2w_2 = \frac{w_0}{D} u(x)u(y)h^4$$

$$\text{or } 20w_2 - 8w_1 + 4w_5 - 16w_3 = \frac{w_0}{D} h^4$$

-(1b)

For Nodal point 3: $x = h, y = 3h$

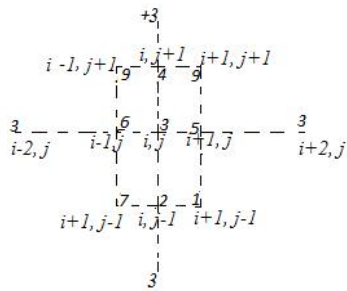


Plate equation in terms of finite difference and singularity function is

$$20w_5 - 16w_3 - 8w_1 + 4w_2 = \frac{w_0}{D} u(x)u(y)h^4 \text{ ----- (1c)}$$

as $u(x) = 1$ & $u(y) = 1$ for all positive values of x .

For nodal point 5: $x = 2h$, $y = 3h$

$$20w_5 - 8(2w_3 + w_9 + w_1) + 2(w_4 + w_{10} + 2w_2) + 2w_6 + w_5 - w_5 = \frac{w_0}{D} h^4$$

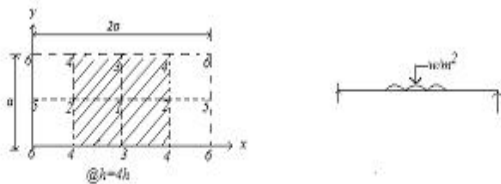
$$20w_5 - 16w_3 - 8w_1 + 4w_2 = \frac{w_0}{D} h^4 \text{ ----- (1d)}$$

Solving the equations (1a), (1b), (1c), (1d) we obtain the deflection w_1, w_2, w_3, w_5 .

Problem 2:

A simply supported rectangular plate of side $2a$ and a , supports an udl partially over an area of intensity w /area. Find the deflection at nodal points.

Fig 7(a) & 7(b)



For nodal point 1:

Conditions are $w_4 = w_6 = w_2 = w_5 = 0$

$x = 2h$, $y = h$

Applying finite difference equation and singularity function

$$20w_1 - 8(2w_2 + 2w_3) + 2(4w_4) + 2w_5 - 2w_1 = w_0 [u(x-h) - u(x-3h)]u(y)$$

$$18w_1 - 16w_2 = \frac{w_0 h^4}{D} \text{ ----- (2a)}$$

For nodal point 2: $x = h$, $y = h$

Applying finite difference equation and singularity function

$$20w_2 - 8w_1 + 2w_3 - 2w_2 = \frac{w_0}{2} [u(x-h) - u(x-3h)]u(y)$$

$$18w_2 - 8w_1 = \frac{w_0}{2} \left(\frac{h^4}{D} \right) \text{ ----- (2b)}$$

The term $1/2$ is multiplied for the following reason.

Load intensity at nodal point 2 can be found as

Let us take an elemental area $p \times p$

$$\text{Total load acting on this area} = w_0 p \left(\frac{p}{2} \right)$$

If the same load is uniformly distributed over $p \times p$, the

intensity of load at nodal point 2 = $\frac{w_0}{2}$. Hence the term

$1/2$ is introduced.

Solving equations (2a) and (2b) we get,

$$w_1 = 0.13 \frac{w_0 h^4}{D}$$

$$w_2 = 0.082 \frac{w_0 h^4}{D}$$

For example Let the size of rectangular plate = $8m \times 4m$

$4h = 8h = 2m$

$w_0 = 500KN / m^2$

$w_1 = 466.3mm$

$t = 50mm$

$E = 2 \times 10^5 N / mm^2$

$\mu = 0.25$

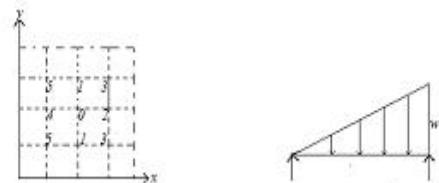
$D = 2.22 \times 10^9 Nmm$

$w_2 = 294.1mm$

Problem 3:

A simply supported R.C.C. square slab of side length $3m$ supports a load of intensity zero at one edge, varying linearly to a value of $8KN/m^2$ at the opposite edge. If the thickness of slab is $12cm$ Poisson's ratio is 0.15 , Modulus of elasticity of concrete is $30,000N/mm^2$. Estimate the maximum deflection in the slab. Adopt 4 grids in each direction.

Fig 7(a) & 7(b)



Applying finite difference with singularity function at nodal points 0, 1, 2, 3, 4 and 5 we have the following equations

At nodal point 0:

$$20z_0 - 16z_1 - 8z_2 - 8z_4 + 4z_5 + 4z_3 = 0.5 \frac{wh^4}{D}$$

At nodal point 1:

$$20z_1 - 8z_0 - 8z_5 - 2z_2 + 2z_4 = 0.5 \frac{wh^4}{D}$$

At nodal point 2:

$$20z_2 - 16z_3 - 8z_0 + 4z_1 - z_2 + z_4 = 0.75 \frac{wh^4}{D}$$

At nodal point 3:

$$20z_3 - 8z_1 - 8z_2 + 2z_0 + z_3 + z_5 = 0.75 \frac{wh^4}{D}$$

At nodal point 4:

$$20z_4 - 16z_3 - 8z_0 + 4z_1 - z_4 + z_2 = 0.25 \frac{wh^4}{D}$$

At nodal point 5:

$$20z_5 - 8z_1 - 8z_4 + 2z_0 + z_5 + z_3 - 2z_5 = 0.25 \frac{wh^4}{D}$$

Solving the above equations we obtain

$$z_0 = 0.002015 \frac{wa^4}{D}$$

$$z_1 = 0.00147 \frac{wa^4}{D}$$

$$z_2 = 0.00163 \frac{wa^4}{D}$$

$$z_3 = 0.0012 \frac{wa^4}{D}$$

$$z_4 = 0.0013 \frac{wa^4}{D}$$

$$z_5 = 0.00093 \frac{wa^4}{D}$$

1.4 DISCUSSION

After the advent of modern methods of plate analysis especially the finite element method the older ones have gone to the background. Levy's and Navier's solutions involve rigorous mathematics, namely partial differential equations even for plates with simple boundary and loading conditions. Even use of algebraic expressions to express loading function instead of trigonometric functions like Fourier series has the disadvantage of handling them while differentiating and in the integration process. To overcome all these difficulties expressing the loading function "q", using the singularity functions are found to be convenient and immensely useful. The finite difference approach to plate problems is convenient for certain type of loading, boundary conditions and stiffness. The method always requires to express "q" as a udl a method becomes difficult and not trackable when multiple concentrated loads are present. When number of mathematical discontinuities occurs due to loading or geometry of stiffness, the number of finite difference equations increase and the use of computers becomes a must to get a solution. Singularity functions useful to express loading, geometry or stiffness, together or separately in a convenient and a useful form. If singularity function are used to express a loading function (q) stiffness function (D) in the finite difference method the potentialities of the finite approach not only gets amplified but also simplifies the method of analysis. The finite method of analysis of plates appears to be the best possible approach to any type of problem of plates. But this method requires use of computers and computer time for plate analysis; Many times software is not available to the structural designers and hence cannot get the solution to this method. In contrast, the use of

singularity function along with the finite difference technique enables one to solve plate problem with any type of loading, boundary and stiffness variation with or without the use of computers in general. A practical designer can find it convenient to analyse involved in finite element method or classical methods. Whenever the mathematical discontinuities occur in structural engineering problems (like due to loading, geometry, and conditions elastic properties etc.) the approach to get a solution becomes difficult. The use of singularity functions in such plates to express mathematical discontinuities in a convenient and useful form is found to be very advantageous. Even though, these functions do not have the properties of continuity or differentiability, in true mathematical sense, at the point of mathematical discontinuity, they are powerful tools in bringing linkage or continuity in elastic equation in structural analysis. Further studies on the application of the proposed technique to the problems of plates of various shapes like circular or triangular plates may open up new vistas in structural analysis.

1.5 CONCLUSIONS

1. A new direction to the solution of plate problems is presented. It is a novel technique for finding elastic functions or evaluating quantities like deflection at a section. The method is simple, less tedious than other methods and is applicable for variable or constant stiffness plates and any type of loading, boundary conditions in general.
2. Use of singularity function like Dirac Delta function, Unit step function, along with finite difference method helps in deriving single expression for elastic function mainly deflection from which other quantities can be calculated. Method is applicable for various types of plates (fixed, simply supported or continuous) having constant, with or without the use of computer help.
3. One of the new directions of analysis of plates presented here involves in using finite difference technique with singularity functions. The solution is fairly accurate and sufficient for the analysis of plates. For more accuracy, the number of division can be increased (in mesh size).
4. Compared to other numerical methods like finite element method which necessarily involves computer use, the proposed method cannot only be handled manually but also amenable to computer applications.
5. Using singularity functions, flexural rigidity (D) and load variation on the plate (q) can be handled with ease. If the variation of above mentioned function (D) and load

- variation (q) cannot be expressed in terms of Fourier series, the use of singularity function can prove to be marvellous innovation in expressing them.
6. It is not essential to one correspondence between the load at nodal point deflection, as in finite difference and finite element approaches. That is, load point and nodal point can be different if singularity functions are used in finite difference method.
 7. The application of singularity functions to the analysis of plates is a contribution of mathematics, in the sense, it may portray a new dimension to the potentiality and application of singularity functions.
 8. Different types of loads on the plate can be analysed without help of principle of superposition or equivalent representation method.

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